Weights in a Serre type conjecture for n-dimensional Galois representations

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I. Serre's Conj. II. A generalization.

III. Some evidence IV. GSp_4.

Notations: p = prime Fix $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

$G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \supseteq G_p$ = decomposition grp of $G_{\mathbb{Q}}$ at p. $\supseteq I_p$

$(N,p)=1$.

(I). $f \in S_k(\Gamma_1(N))$ $k \geq 2$
eigen form

$T_\ell f = a_\ell f$, $S_\ell f = b_\ell f$ $\ell \nmid N$.

Deligne: $\exists$ $\rho_{f,p}: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ cont. s.s.

(i) unramified outside pN & $\ell \nmid pN$,

$\det(1-\rho(\mathrm{Frob}_\ell)X) = 1 - \bar{a}_\ell X + \ell \bar{b}_\ell X^2$

(ii) $\rho_{f,p}$ odd $\rho_{f,p}(\text{cx conj}) \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

Conj (Serre): If $\rho: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ is odd & irr.

then $\rho \cong \rho_{f,p}$ for some eigen form f of level $N(\rho)$? and wt $k(\rho)$? ≥ 2

$\{\rho|_{I_\ell} : \ell \neq p\}$ $\{\rho|_{I_p}\}$

Proved finally by Khare-Wintenberger (+ Kisin) '06/'07

SC $\Rightarrow$ Strong Artin Conj for odd, 2-dim Gal rep'n.

Serre wts. : irred rep'ns F of $GL_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$

$$F(a,b) \cong \mathrm{Sym}^{a-b}\, \overline{\mathbb{F}}_p^2 \otimes \det^b \qquad 0 \le a-b \le p-1.$$

Connection: wt $k \geq 2$,

$$\left\{ \begin{array}{l} \text{cohomology class} \\ \text{coeff} = \mathrm{Sym}^{k-2}\, \overline{\mathbb{Q}}^2 \end{array} \right\} \overset{\text{mod } p}{\rightsquigarrow} \bigcup \left\{ \begin{array}{l} \text{cohom. class} \\ \text{coeff} = F \end{array} \right\}$$

Eichler-Shimura $\qquad F \in JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2)$ as rep'n of $GL_2(\mathbb{F}_p)$. something related to reduction mod p

$H^1(\Gamma(N), -)$

$\rightsquigarrow$ dictionary $\quad k^?(\rho) \longleftrightarrow W^?(\rho) =$ set of predicted Serre wts.

Example: If $\rho|_{I_p} \sim \begin{pmatrix} \omega^{k-1} & \\ & 1 \end{pmatrix}$, $\qquad k^?(\rho) = k$

$2 < k < p-1$

$\rho|_{I_p} \otimes \omega^{k-1} \sim \begin{pmatrix} \omega^{p-k} & \\ & 1 \end{pmatrix} \qquad k^?(\rho \otimes \omega^{1-k}) = p+1-k$

$W^?(\rho) = \{ \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2 = F(k-2,0), F(p-1,k-1) \}$.

Rmks: (1) For generic tame $\rho|_{I_p}$, $\#W^?(\rho) = 2$ $\qquad$ (tame $\Leftrightarrow$ s.s. ?)

(2) $W^?(\rho|_{I_p}) \subset W^?(\rho|_{I_p}^{ss})$

(3) In above ex. $W^?(\rho) =$ constituents of red'n mod p of a cusp. rep'n of $GL_2(\mathbb{F}_p)$.

(II) $n \geq 2$, $(N,p)=1$

$$\Gamma_1(N) = \left\{ \gamma \in SL_n(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} & * & \\ 0 & \cdots & 0\ 1 \end{pmatrix} \bmod N \right\}.$$

Serre wts: irr. rep'n F of $GL_n(\mathbb{F}_p) / \overline{\mathbb{F}}_p$

$F(a_1, \dots, a_n)$, $\qquad 0 \le a_i - a_{i+1} \le p-1$

+ regular if $\quad \dots\dots < p-1 \quad \forall i$.

$\alpha \in H^e(\Gamma_1(N), F)$ Hecke eigenvector $\longleftrightarrow$ $\rho: G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$.

ρ $\boxed{\text{modular of wt } F}$ if

"Hecke polynomial" of α = char poly of $\rho(\mathrm{Frob}_\ell)$ $\forall \ell \nmid pN$

for some eigenvec. α, $e \geq 0$, $(N,p)=1$.

$W(\rho) = \{$ regular F : ρ modular of wt $F\}$

$\rho(\text{cx conj}) \sim \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \ddots \end{pmatrix}$

Conj (H) If $\rho: G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ odd, irred. and $\rho|_{I_p}$ tame, then $W(\rho) = W^?(\rho) := \mathcal{R}(JH(\overline{V(\rho|_{I_p})}))$

① $\mathcal{R}: \{\text{Serre wts}\} \to \{\text{regular Serre's wts}\}$

If $F(\mu)$ regular, $\mathcal{R}(F(\mu)) = F(w_0 \cdot (\mu - p\tilde{\rho}))$

$(\mu \in X(T))$ — w_0: longest Weyl elt. — $\tilde{\rho}$: $(n-1, n-2, \ldots, 1, 0)$

② $G = \hat{G} = GL_n$

$V: \left\{ \begin{array}{l} \tau: I_p \to \hat{G}(\overline{\mathbb{F}}_p) \\ \text{that extends to } G_p \end{array} \right\} / \cong \longleftrightarrow \left\{ \begin{array}{l} \text{Deligne-Lusztig rep'ns of} \\ G(\mathbb{F}_p) \text{ over } \overline{\mathbb{Q}}_p \end{array} \right\} / \cong$

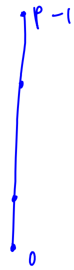
(duality of rational max. tori in $G/\mathbb{F}_p$)

Ex $(n=3)$, $\rho|_{I_p} \sim \begin{pmatrix} \omega^i & & \\ & \omega^j & \\ & & \omega^k \end{pmatrix} \to V(\rho|_{I_p})$ principal series

$\rho|_{I_p} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2 m} \end{pmatrix} \Rightarrow V(\quad)$ cuspidal

<u>Combinatorics</u> of $W^?(\rho|_{I_p})$: Jantzen '81

$n=2$:



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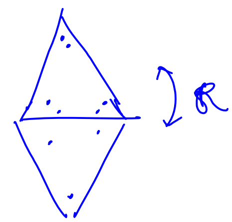
cuspidal
$\updownarrow$
principal

$n=3$:



9.

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Earlier Conj of Ash, Doud, Pollack, Sinnott $(n=3)$.

- combinatorial
- <9 wts predicted in many cases.

III. Evidence

(i) $n=3$: Computations (Doud, Pollack).
$(\operatorname{Im}\rho = S_4,\ 3\cdot A_6, \dots)$

(ii) $n=4$ $\quad F=F(a,b,c,d)$ $\quad a+d=b+c$ $\quad$ generic $\quad a-d<p$

Using automorphic inductions of Hecke characters $\quad$ K CM, non-Gal. $\quad K \,|_4\, \mathbb{Q}$

obtaining many odd, irred ρ with $\rho|_{I_p}$ tame s.t.

$F \in W(\rho) \cap W^?(\rho|_{I_p})$ $\qquad (F, \rho|_{I_p})$: 8 out of 24

automorphic induction...

$$\begin{tikzcd} G_F \arrow[r] \arrow[dr, dashed] & G_{\mathbb{Q}} \arrow[d] \\ & GL_2(\mathbb{F}_p) \end{tikzcd}$$

(iii) Conj: of Buzzard-Diamond-Jarvis

$\rho: G_F \to GL_2(\overline{\mathbb{F}}_p)$ irred., totally odd $\qquad$ F tot real, unramified at p over $\mathbb{Q}$

If ρ tame at p, predicted wts, predicted Serre wts compatible w/ above conj.

Rmk (Diamond): Compatibility with our apply R ... $\begin{pmatrix} \omega^i & \\ & \omega^j \end{pmatrix}$ ...

IV w/ Tilouine

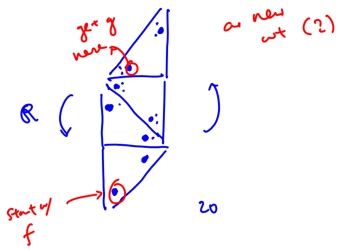
$G = \hat{G} = GSp_4$ $\qquad J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ preserve up to scalar

$\rho: G_{\mathbb{Q}} \to GSp_4(\overline{\mathbb{F}}_p)$

- irred
- "odd" $\qquad \rho(\text{cx conj}) \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\quad (\Leftrightarrow \overset{\text{det.}}{\downarrow}\ \nu\circ\rho(\text{cx. conj}) = -1)$
- ρ modular of wt F: $\qquad \rho \subseteq H^*_{et}(X_{\overline{\mathbb{Q}}}, F)$ $\quad$ (X: Shimura variety level prime to p)

Get a conj. as above if
- ρ tame at p
- regular wts.

$\rho|_{I_p}$ tame & generic



Thm: (Tilouine) f cusp. holomorphic eigenform of wt. (k_1, k_2), $k_1 \geq k_2 \geq 3$

$k_1+k_2-3 < p-1$

(1) f ordinary

(2) $\bar{\rho}_{f,p}\big|_{I_p} \sim \begin{pmatrix} \omega^{k_1+k_2-3} & 0 & * & * \\ & \omega^{k_1-1} & * & * \\ & & \omega^{k_2-2} & 0 \\ & & & 1 \end{pmatrix}$

So $\bar{\rho}_{f,p} \otimes \omega^{2-k_2}\big|_{I_p} \sim \begin{pmatrix} \omega^{k_1'+k_2'-3} & 0 & * & * \\ & \omega^{k_1'-1} & * & * \\ & & \omega^{k_2'-2} & 0 \\ & & & 1 \end{pmatrix}$

$k_1' = k_1 + p - 1$

$k_2' = p + 3 - k_2$

$\bar{\rho} \otimes \omega^{2-k_2} \xrightarrow{\text{sub-quotient}} H^*_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \underbrace{\omega(\ldots)}_{\text{weyl module}})$

(3) $\bar{\rho}_{f,p}$ large-image

(4) $\rho_{f,p}$ min. ramified lift of $\bar{\rho}_{f,p}$, level N square free

then $\exists$ g cusp. holomorphic eigenform of (k_1', k_2') s.t. $\bar{\rho}_{g,p} \cong \bar{\rho}_{f,p} \otimes \omega^{2-k_2}$

(Pf uses Faltings' comparison thm & BGG cx.)